POYNTING THEOREM & POYNTING VECTOR

Poynting theorem: For obtaining the power in a uniform plane wave, in 1884 John H. poynting develop a power theorem for electromagnetic field by using Maxwell's equation, which is known as poynthing theorem.

Statement of poynting theorem:- At any point in a uniform plane wave, the cross product of electric field intensity $\vec{E}$ and magnetic field intensity $\vec{H}$ gives measure of rate of energy flow per unit area at that point.

Mathematically, $\vec{S} = \vec{E} \times \vec{H}$ = poynting vector

which is given as

$$\underset{\underset{\text{Rate of energy flow}}{\downarrow}}{\oint_A (\vec{E} \times \vec{H}) \cdot d\vec{A}} = -\underset{\underset{\text{power loss}}{\downarrow}}{\int_V (\vec{E} \cdot \vec{J})\, dv} - \underset{\underset{\text{Rate of decrease in EM energy}}{\downarrow}}{\frac{\delta}{\delta t} \int \left(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right) dv} \quad \text{——①}$$

It is known as poynting theorem.

proof of poynting theorem:- poynting theorem can be proved by using Maxwell's equation.

From Maxwell's third equation,

$$\vec{\nabla} \times \vec{E} = -\frac{\delta \vec{B}}{\delta t} \Rightarrow \vec{\nabla} \times \vec{E} = -\frac{\delta}{\delta t}(\mu \vec{H}) \quad \because \vec{B} = \mu \vec{H}$$

$$\Rightarrow \vec{\nabla} \times \vec{E} = -\mu \cdot \frac{\delta \vec{H}}{\delta t}$$

Taking dot product both sides with $\vec{H}$

$$\vec{H} \cdot (\vec{\nabla} \times \vec{E}) = -\mu \left(\vec{H} \cdot \frac{\delta \vec{H}}{\delta t}\right) \quad \text{——②}$$

From Maxwell's fourth equation,

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\delta \vec{D}}{\delta t} \Rightarrow \vec{\nabla} \times \vec{H} = \vec{J} + \frac{\delta(\varepsilon \vec{E})}{\delta t} \quad \because \vec{D} = \varepsilon \vec{E}$$

$$\Rightarrow \vec{\nabla} \times \vec{H} = \vec{J} + \varepsilon \frac{\delta \vec{E}}{\delta t}$$

Taking dot product bothsides with $\vec{E}$.

② Electromagnetic wave: Dr. Md. NAIYAR PERWEZ

$$\vec{E}\cdot(\vec{\nabla}\times\vec{H}) = \vec{E}\cdot\vec{J} + \varepsilon\left(\vec{E}\cdot\frac{\delta\vec{E}}{\delta t}\right) \text{ ——— ③}$$

Now $$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}(\vec{\nabla}\times\vec{H}) \text{ ——— ④}$$

[using identity $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}(\vec{A}\times\vec{B}) - \vec{B}(\vec{A}\times\vec{C})$]

Now Using equⁿˢ ② and ③ in equⁿ ④, we get

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\mu\left(\vec{H}\cdot\frac{\delta\vec{H}}{\delta t}\right) - \vec{E}\cdot\hat{J} - \varepsilon\left(\vec{E}\cdot\frac{\delta\vec{E}}{\delta t}\right)$$

$$\Rightarrow \vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\vec{E}\cdot\vec{J} - \left[\varepsilon\left(\vec{E}\cdot\frac{\delta\vec{E}}{\delta t}\right) + \mu\left(\vec{H}\cdot\frac{\delta\vec{H}}{\delta t}\right)\right] \text{ — ⑤}$$

Now $$\frac{\delta(\vec{H}\cdot\vec{H})}{\delta t} = \vec{H}\cdot\frac{\delta\vec{H}}{\delta t} + \vec{H}\cdot\frac{\delta\vec{H}}{\delta t} \qquad \left(\because \frac{\delta(xy)}{\delta t} = x\frac{\delta y}{\delta t} + y\frac{\delta x}{\delta t}\right)$$

$$\Rightarrow \frac{\delta H^2}{\delta t} = 2\vec{H}\cdot\frac{\delta\vec{H}}{\delta t} \qquad \because \vec{H}\cdot\vec{H} = H^2$$

$$\Rightarrow \vec{H}\cdot\frac{\delta\vec{H}}{\delta t} = \frac{1}{2}\cdot\frac{\delta H^2}{\delta t} \text{ ——— ⑥}$$

Similarly $$\vec{E}\cdot\frac{\delta\vec{E}}{\delta t} = \frac{1}{2}\cdot\frac{\delta E^2}{\delta t} \text{ ——— ⑦}$$

Using equⁿˢ ⑥ and ⑦ in equⁿ ⑤, we get

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\vec{E}\cdot\vec{J} - \left[\varepsilon\cdot\frac{1}{2}\frac{\delta E^2}{\delta t} + \mu\cdot\frac{1}{2}\frac{\delta H^2}{\delta t}\right]$$

$$\Rightarrow \vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\vec{E}\cdot\vec{J} - \frac{\delta}{\delta t}\left[\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right] \text{ ——— ⑧}$$

Integrating equⁿ ⑧ over the volume V enclosed by a surface of area A.

$$\int_V \vec{\nabla}\cdot(\vec{E}\times\vec{H})\,dV = -\int_V(\vec{E}\cdot\vec{J})\,dV - \frac{\delta}{\delta t}\int\left(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right)dV$$

But from Gauss' divergence theorem, $\int_V \vec{\nabla}\cdot(\vec{E}\times\vec{H})\,dV = \oint_A(\vec{E}\times\vec{H})\cdot d\vec{A}$

so $$\boxed{\oint_A(\vec{E}\times\vec{H})\cdot d\vec{A} = -\int_V(\vec{E}\cdot\vec{J})\,dV - \frac{\delta}{\delta t}\int\left(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right)dV} \text{ — ①}$$

It is poynting theorem. proved.

In free space, $\vec{J} = 0$, $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$

Poynting theorem in free space will be

$$\boxed{\oint_A (\vec{E}\times\vec{H})\cdot d\vec{A} = -\frac{\delta}{\delta t}\int\left(\frac{\epsilon_0 E^2}{2}+\frac{\mu_0 H^2}{2}\right)dV}$$

Poynting vector :- $\vec{E}\times\vec{H} = \vec{S}$ = poynting vector.

Cross product of electric field intensity $\vec{E}$ and magnetic field intensity $\vec{H}$ at any point in a uniform plane wave, is known as poynting vector. It is denoted by $\vec{S}$.

Poynting vector $\vec{S} = \vec{E}\times\vec{H}$ gives the measure of rate of energy flow per unit surface area at a point in a uniform plane wave.

$$\vec{S} = \vec{E}\times\vec{H} = \frac{\text{Rate of energy flow}}{\text{Surface area}}$$

S.I. unit of poynting vector = watt/m^2

Poynting vector $\vec{S} = \vec{E}\times\vec{H} = \vec{E}\times\frac{\vec{B}}{\mu_0}$ $\quad \because \vec{H} = \frac{\vec{B}}{\mu_0}$ in free space

$$\Rightarrow \vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})$$

$$\Rightarrow \vec{S} = c^2\epsilon_0(\vec{E}\times\vec{B}) \qquad \because c^2 = \frac{1}{\mu_0\epsilon_0}$$

Thus poynting vector $\boxed{\vec{S} = \vec{E}\times\vec{H} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = c^2\epsilon_0(\vec{E}\times\vec{B})}$

Ques:- State and prove poynting theorem for the rate of flow of energy in electromagnetic field. What is poynting vector?